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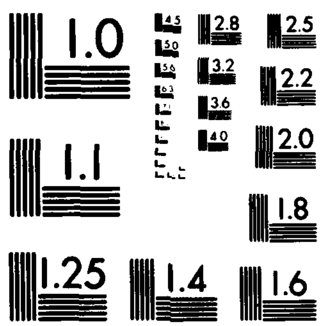

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WE PRESENT A GENERAL APPROXIMATION SCHEME FOR AUTONOMOUS FDE's WITH LIPSCHITZEAN RIGHT-HAND SIDE. OUR APPROACH IS BASED ON APPROXIMATION RESULTS IN SEMIGROUP THEORY. THE RESULTS ARE APPLIED TO NONLINEAR AUTONOMOUS FDE's IN THE STATE SPACE $\mathbb{R}^n \times L^2$ AND TO LINEAR AUTONOMOUS FDE's OF NEUTRAL TYPE IN THE STATE SPACES $\mathbb{R}^n \times L^2$ and $W^{1,2}$.

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AN APPROXIMATION SCHEME FOR DELAY EQUATIONS +

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AN APPROXIMATION SCHEME FOR DELAY EQUATIONS

by

F. Kappel

ABSTRACT:

We present a general approximation scheme for autonomous FDE's with Lipschitzean right-hand side. Our approach is based on approximation results in semigroup theory. The results are applied to nonlinear autonomous FDE's in the state space $\mathbb{R}^n \times L^2$ and to linear autonomous FDE's of neutral type in the state spaces $\mathbb{R}^n \times L^2$ and $W^{1,2}$.

P-1

1. Introduction.

In recent years one can see considerable interest in approximation of delay systems by ordinary differential equations. Results of this type proved to be very useful for the numerical solution of optimal control problems and identification problems (see [2],[5], [6]). A very successful approach uses abstract approximation results in semigroup theory and applies these to the semigroups associated with autonomous delay systems. This approach goes back to [4] (see also [5]) where control problems involving linear autonomous functional-differential equations and approximation of the state by step functions are considered. A considerable improvement over the scheme developed in [5] was obtained in [7] where for the same class of delay systems a scheme is developed which allows approximation of the state by spline functions. In this paper we give an abstract formulation of the scheme developed in [7] which can be applied to autonomous delay systems with globally Lipschitzean right-hand side. We also indicate the application of the linear version of the scheme to autonomous neutral delay systems. Since by lack of space we cannot give a detailed discussion of the relevant literature, we include a rather complete list of papers in the references ([1,3,5,7,13,14,15, 16,17,18,19,20,22,23,24]) and refer to the discussions given there (see especially Section 5 of [5]).

2. An Approximation Scheme for Semigroups of Nonlinear Transformations.

Throughout this section X will be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. A family $T(t)$, $t \geq 0$, of globally Lipschitzean operators $X \to X$ is called a semigroup of type ω, $\omega \in \mathbb{R}$, if the following properties are satisfied:

(i) $T(0) = I$.

(ii) $T(t+s) = T(t)T(s)$ for all $t,s \geq 0$.

(iii) For any $x \in X$ the map defined by $t \to T(t)x$ is continuous on $[0,\infty)$.

(iv) $|T(t)x - T(t)y| \leq e^{\omega t}|x-y|$ for all $x,y \in X$ and $t \geq 0$.

If for a not necessarily single-valued operator A on X, the operator $(I-\lambda A)^{-1}$ is single-valued and defined on X for λ sufficiently small, we say that $T(\cdot)$ is generated by A if for any $x \in X$

$$T(t)x = \lim_{n\to\infty} (I-\frac{t}{n} A)^{-n}x, \quad t \geq 0,$$

uniformly with respect to t in bounded intervals. Of fundamental importance is the generation theorem by Crandall-Liggett ([12]):

Let A be a densely defined operator on X. If $A - \omega I$ is dissipative for some $\omega \in \mathbb{R}$ and if range $(I-\lambda A) = X$ for sufficiently small λ, then A generates a semigroup of type ω on X.

If A is single-valued then $A - \omega I$ is dissipative if and only if $(Ax-Ay,x-y) \leq \omega|x-y|^2$ for all $x,y \in \mathrm{dom}\, A$. But note, that the Crandall-Liggett Theorem is valid in general Banach spaces. Fundamental for our approach is the following approximation theorem for semigroups of type ω:

Theorem 1 ([10]). Let A_N, $N = 1,2,\ldots,$ and A be single-valued operators on X such that $\mathrm{dom}\, A_N \supset \mathrm{dom}\, A$ for all N and $\overline{\mathrm{dom}\, A} = X$. Assume that the

following conditions are satisfied:

(i) There exists a $\lambda_0 > 0$ such that

$$\text{range } (I-\lambda A) = \text{range } (I-\lambda A_N) = X$$

for $N = 1,2,\ldots$ and all $\lambda \in (0,\lambda_0)$.

(ii) There exist real constants ω_N, $N = 1,2,\ldots$, and ω such that the sequence $\{\omega_N\}$ is bounded above and $A_N - \omega_N I$ and $A - \omega I$ are dissipative.

(iii) There exists a subset G of dom A such that $\overline{(I-\lambda A)G} = X$ for λ sufficiently small and

$$A_N x \to Ax \quad \text{as} \quad N \to \infty$$

for all $x \in G$.

Then A_N and A generate semigroups $T_N(\cdot)$ and $T(\cdot)$ of type ω_N and ω, respectively, and for all $x \in X$

$$\lim_{N\to\infty} T_N(t)x = T(t)x$$

uniformly on bounded t-intervals.

This theorem follows immediately from Theorems 4.1 and 3.1 in [10] and is also true in general Banach spaces.

<u>Definition 1.</u> Assume that A is an operator on X. A sequence $\{X_N,P_N,A_N\}$, $N = 1,2,\ldots$, is called an approximation scheme related to A if

(i) X_N are finite dimensional linear subspaces of dom A, $N = 1,2,\ldots$.

(ii) $P_N: X \to X_N$ is the orthogonal projection onto X_N, $N = 1,2,\ldots$.

(iii) $A_N = P_N A P_N$, $N = 1,2,\ldots$.

Condition (i) in this definition is the most restrictive one. It could mean that only the trivial approximation scheme $(0,0,0)$ for all N is related to A. If A generates a semigroup of some type ω it is of interest to know when the operators A_N generate semigroups $T_N(\cdot)$ converging to $T(\cdot)$.

<u>Theorem 2.</u> Let A be a densely defined single-valued operator on X such that $A - \omega I$ is dissipative for some $\omega \in \mathbb{R}$ and range $(I-\lambda A) = X$ for λ sufficiently small. Furthermore, let $\{X_N, P_N, A_N\}$ be an approximation scheme related to A. Assume that the following conditions are satisfied:

(i) $\lim\limits_{N\to\infty} P_N x = x$ for all $x \in X$.

(ii) There exists a set $G \subset \text{dom } A$ such that $\overline{(I-\lambda A)G} = X$ for λ sufficiently small and

$$\lim_{N\to\infty} AP_N x = Ax \quad \text{for all} \quad x \in G.$$

(iii) For each N there exists a $\lambda_N > 0$ such that

$$\text{range } (I-\lambda A_N) = X$$

for all $\lambda \in (0,\lambda_N)$.

Then the following is true:

a) A_N is continuous on X and generates a semigroup of type ω on X, $N = 1,2,\ldots$.

b) $T_N(t)X_N \subset X_N$ for $t \geq 0$, $N = 1,2,\ldots$.

c) $\lim\limits_{N\to\infty} T_N(t)P_N x = T(t)x$ for all $x \in X$ uniformly on bounded t-intervals, where $T(\cdot)$ is the semigroup generated by A.

<u>Proof.</u> Condition (ii) of Theorem 1 is satisfied with $\omega_N = \omega$. This follows from

$$(A_N x - A_N y, x-y) = (P_N[AP_N x - AP_N y], x-y) =$$

$$(AP_N x - AP_N y, P_N x - P_N y) \leq \omega |P_N x - P_N y|^2 \leq \omega |x-y|^2$$

for all x,y in X. Here we have used $P_N^* = P_N$. For $x \in G$ we have

$$|A_N x - Ax| \leq |P_N AP_N x - P_N Ax| + |P_N Ax - Ax|$$

$$\leq |AP_N x - Ax| + |P_N Ax - Ax|.$$

By conditions (ii) and (i) both terms on the right-hand side vanish as $N \to \infty$. Thus, also condition (iii) of Theorem 1 is satisfied. It only remains to prove that range $(I-\lambda A_N) = X$ for $N = 1,2,\ldots$ and all $\lambda \in (0,\lambda_0)$, λ_0 a positive number not dependent on N, and that A_N is continuous on X.

We have range $(I-\lambda(A_N-\omega I))$ = range $(I-\frac{\lambda}{1+\lambda\omega} A_N) = X$ for all λ with $0 < \lambda(1+\lambda\omega)^{-1} < \lambda_N$. Since $A_N - \omega I$ is dissipative, we immediately get (cf. [8; p.73] or [9; p.23]) range $(I-\lambda(A_N-\omega I)) = X$ for all $\lambda > 0$, i.e., range $(I-\lambda A_N) = X$ for $\lambda \in (0,\frac{1}{\omega})$. This proves condition (i) of Theorem 1 with $\lambda_0 = \frac{1}{\omega}$. Since $A_N - \omega I$ is m-dissipative and defined on all of X, it is also demicontinuous, i.e., continuous from X with the norm topology into X with the weak topology. But $A_N X \subset X_N$ and dom $X_N < \infty$ imply that A_N is continuous. Finally, conclusion b) follows from $A_N X_N \subset X_N$.

<u>Corollary 1.</u> If A is linear, then the conclusion of Theorem 2 holds without assuming condition (iii) explicitly. Moreover, $T_N(t)$ is given by $T_N(t) = \exp A_N t$.

Proof. The conditions on A imply (also in the nonlinear case) that A is closed (cf. [8; p.75]). Then also AP_N is closed and defined on all of X. Therefore, by the Closed-Graph Theorem AP_N is a bounded linear operator. The rest of the proof is clear.

In the linear case we also can replace the condition "range $(I-\lambda A) = X$ for λ sufficiently small" by "A is the infinitesimal generator of a C_o-semigroup". This is a consequence of the Lumer-Phillips Theorem (see for instance [21; p.16]).

The following corollary covers a situation taylored for delay systems:

Corollary 2. Let A_0 be a closed linear operator with $\operatorname{dom} A_0 = X$ and A_1 be an operator with $\operatorname{dom} A_1 \supset \operatorname{dom} A_0$ such that

$$|A_1x-A_1y| \le L\,\|x-y\|$$

for all $x,y \in \operatorname{dom} A_0$, where $L > 0$ and $\|\cdot\|$ is a norm on $\operatorname{dom} A_0$. Then condition (iii) of Theorem 2 holds for any approximation scheme $\{X_N,P_N,A_N\}$ related to $A = A_0 + A_1$.

Proof. The conditions on A_0 imply that $A_{N,0} = P_NA_0P_N$ is a bounded linear operator, $\alpha_N = |A_{N,0}|$. We have to prove that $(I-\lambda A_N)y = x$ has a solution y for any $x \in X$ provided λ is sufficiently small. This equation is equivalent to

$$y = \lambda(I-\lambda A_{N,0})^{-1}P_NA_1P_Ny + (I-\lambda A_{N,0})^{-1}x =: h(y)$$

if $\lambda \in (0,\alpha_N^{-1})$. Then for all $y_1,y_2 \in X$

$$|h(y_1)-h(y_2)| \le \lambda(1-\alpha_N\lambda)^{-1}|A_1P_Ny_1-A_1P_Ny_2|$$
$$\le \lambda L(1-\alpha_N\lambda)^{-1}\|P_Ny_1-P_Ny_2\| \le \sigma_N\lambda L(1-\alpha_N\lambda)^{-1}|y_1-y_2|.$$

Here we have used the fact that two norms on a finite dimensional space are equivalent. Note, that P_Ny_1 and P_Ny_2 are in X_N. The estimate shows that h is a contraction for λ sufficiently small.

Using, for instance, Theorem II of [12] and the fact that the Peano-Existence Theorem is valid in finite dimensional spaces, we see that $u(t) = T_N(t)P_Nx$ for any $x \in X$ is the solution of

$$\dot{u}(t) = A_Nu(t), \quad t \ge 0,$$
$$u(0) = P_Nx,$$

which is an initial value problem for an ordinary differential equation on X_N. For calculations one has to choose a basis for X_N and to represent A_N and P_Nx with respect to this basis (see [7] and [15] for more details).

3. Applications to Functional Differential Equations.

a) Nonlinear Equations in $\mathbb{R}^n \times L^2$

We consider autonomous equations

$$\dot{x}(t) = f(x_t), \quad t \ge 0, \tag{3.1}$$

with initial data

$$x(0) = \eta \in \mathbb{R}^n, \; x(s) = \phi(s) \text{ a.e. on } [-r,0], \tag{3.2}$$

$r > 0$, where $\phi \in L^2(-r,0;\mathbb{R}^n)$. For a function $x: [-r,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, x_t is defined by $x_t(s) = x(t+s)$, $s \in [-r,0]$. A solution of (3.1), (3.2) is a function x on an interval $[-r,\alpha)$, $\alpha > 0$, such that (3.2) holds and $x(t) = \eta + \int_0^t f(x_s)ds$ for $t \in [0,\alpha)$. We assume that f is a map $\mathscr{L}^2(-r,0;\mathbb{R}^n) \to \mathbb{R}^n$ ($\mathscr{L}^2$ is the linear space of square integrable functions) satisfying

(h1) For any function x in $L^2(-r,\alpha;\mathbb{R}^n)$, $\alpha > 0$, which is continuous on $[0,\alpha)$ $t \to f(x_t)$ uniquely defines a map in $L^1(0,\alpha;\mathbb{R}^n)$.

(h2) There exist constants $L > 0$ and r_j, $j = 0,\dots,m$, $0 = r_0 < r_1 < \dots < r_m = r$, such that for any ϕ,ψ in $\mathscr{L}^2(-r,0;\mathbb{R}^n)$

$$|f(\phi) - f(\psi)| \le L\Big(\sum_{j=0}^{m} |\phi(-r_j)-\psi(-r_j)| + |\phi-\psi|_{L^2}\Big).$$

Under conditions (h1), (h2) problem (3.1), (3.2) has a unique solution $x(t;\eta,\phi)$ on $[-r,\infty)$ for any $(\eta,\phi) \in X = \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$. The family of operators $T(\cdot)$ defined by $T(t)(\eta,\phi) = (x(t;\eta,\phi),x_t(\eta,\phi))$, $t \ge 0$, $(\eta,\phi) \in X$, is a strongly continuous semigroup of globally Lipschitzean operators on X (cf. [17]). X is a Hilbert space with norm $|(\eta,\phi)|_g = (|\eta|^2 + \int_{-r}^0 |\phi(s)|^2 g(s)ds)^{1/2}$ and corresponding inner product, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and the weighting function g is defined by

$$g(s) = m-j+1 \quad \text{for} \quad s \in (-r_j,-r_{j-1}), \quad j = 1,\dots,m.$$

Define the operator A by

$$\operatorname{dom} A = \{(\phi(0),\phi) \mid \phi \in W^{1,2}(-r,0;\mathbb{R}^n)\},$$
$$A(\phi(0),\phi) = (f(\phi),\dot{\phi}), \quad \phi \in W^{1,2}(-r,0;\mathbb{R}^n).$$

Theorem 3. Let $\{X_N, P_N, A_N\}$ be an approximation scheme related to A such that

(i) $\lim_{N\to\infty} P_N x = x$ for all $x \in X$.

(ii) There exists an integer $k \geq 1$ such that for $\phi \in C^k(-r,0;\mathbb{R}^n)$

$$\phi_N(0) \to \phi(0) \quad \text{and} \quad \dot{\phi}_N \to \dot{\phi} \quad \text{in} \quad L^2(-r,0;\mathbb{R}^n)$$

as $N \to \infty$. Here $\phi_N \in W^{1,2}(-r,0;\mathbb{R}^n)$ is defined by

$$P_N(\phi(0),\phi) = (\phi_N(0),\phi_N) \in \text{dom } A.$$

Then, for each N, A_N generates a semigroup $T_N(\cdot)$ on X such that for all $x \in X$

$$\lim_{N\to\infty} T_N(t)P_N x = T(t)x$$

uniformly on bounded t-intervals.

Indication of proof -(a complete representation of the results concerning equation (3.1) will appear elsewhere [15]).

Step 1: We have $\overline{\text{dom } A} = X$. For the set G in condition (ii) of Theorem 2 we take $G = \{(\phi(0),\phi) \mid \phi \in C^k(-r,0;\mathbb{R}^n)\}$. Using (h2) one shows that there exists an $\omega \in \mathbb{R}$ such that $A - \omega I$ is dissipative in X. The idea to introduce a weighting function is due to G.F. Webb [25] and is essential here. In order to verify the range condition on A one observes that $(I-\lambda A)(\phi(0),\phi) = (\eta,\psi) \in X$ is equivalent to $\phi(s) = e^{s/\lambda}\phi(0) - \frac{1}{\lambda}\int_{-r}^{0} e^{(s-\tau)/\lambda}\psi(\tau)d\tau$, $s \in [-r,0]$, and $\phi(0) = h(\phi(0))$, where $h(a) = \eta + \lambda f(e^{\cdot/\lambda}a - \frac{1}{\lambda}\int_{-r}^{\cdot} e^{(\cdot-\tau)/\lambda}\psi(\tau)d\tau)$ for $a \in \mathbb{R}^n$. Using (h2) one shows that h is a contraction on $\mathbb{R}^n$ for λ sufficiently small.

Step 2: By definition of A we see that $AP_N x \to Ax$ in D is equivalent to $f(\phi_N) \to f(\phi)$ and $\dot{\phi}_N \to \dot{\phi}$ in $L^2(-r,0;\mathbb{R}^n)$. But $f(\phi_N) \to f(\phi)$ follows from (ii) and (h2). Therefore, condition (ii) of Theorem 2 is satisfied.

Step 3: In order to apply Corollary 2 we write $A = A_0 + A_1$, where $A_0(\phi(0),\phi) = (0,\dot{\phi})$ and $A_1(\phi(0),\phi) = (f(\phi),0)$ for $\phi \in W^{1,2}(-r,0;\mathbb{R}^n)$. Then A_0 is a densely defined and closed linear operator (see [17]). (h2) implies that A_1 satisfies the condition in Corollary 2 with $\|(\phi(0),\phi)\| = \sup_{-r\le s\le 0} |\phi(s)|$. Thus, condition (iii) of Theorem 2 is satisfied.

Step 4: The semigroup generated by A is indeed $T(\cdot)$ (cf. [17]).

The conclusion of Theorem 3 in the case of spline approximation was also obtained by H.T. Banks in [3] by a different method under the additional assumption that f is differentiable.

b) Equations of Neutral Type in $\mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$

Let D and L be bounded linear functionals $C(-r,0;\mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$D(\phi) = \phi(0) - \sum_{j=1}^{m} B_j\phi(-r_j) - \int_{-r}^{0} B(s)\phi(s)ds,$$

$$L(\phi) = \sum_{j=0}^{m} A_j\phi(-r_j) + \int_{-r}^{0} A(s)\phi(s)ds,$$

where $0 = r_0 < r_1 < \ldots < r_m = r$. A_j, B_j are $n \times n$ matrices and $A(\cdot), B(\cdot)$ are in $L^2(-r,0;\mathbb{R}^{n\times n})$.

As before we take $X = \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$. Following an idea given in [11] we define the operator A by

$$\text{dom } A = \{(D(\phi),\phi) \mid \phi \in W^{1,2}(-r,0;\mathbb{R}^n)\},$$

$$A(D(\phi),\phi) = (L(\phi),\phi), \quad \phi \in W^{1,2}(-r,0;\mathbb{R}^n).$$

X will be supplied with the norm $|(\eta,\phi)|_g$, where the weighting function g now is also dependent on D:

$$g(s) = m-j+1+\alpha_j(s+r_{j-1}) \quad \text{for} \quad s \in (-r_j,-r_{j-1}),\ j = 1,\dots,m,$$

$$\alpha_j \leq -\frac{m(m+2)}{r_j-r_{j-1}} |B_j|^2 , \quad j = 1,\dots,m.$$

Theorem 4.

a) A generates a C_o-semigroup $T(\cdot)$ of bounded linear operators on X.

b) Let $\{X_N,P_N,A_N\}$ be an approximation scheme related to A satisfying:

(i) $\lim_{N\to\infty} P_N x = x$ for all $x \in X$.

(ii) There exists an integer $k \geq 1$ such that for $\phi \in C^k(-r,0;\mathbb{R}^n)$ $\phi_N(0) \to \phi(0)$ and $\dot{\phi}_N \to \dot{\phi}$ in $L^2(-r,0;\mathbb{R}^n)$ as $N \to \infty$, where $\phi^N \in W^{1,2}(-r,0;\mathbb{R}^n)$ is defined by

$$P_N(D(\phi),\phi) = (D(\phi_N),\phi_N).$$

Then, for each N, A_N is a bounded linear operator which generates the C_o-semigroup $T_N(t) = \exp A_N t$, $t \geq 0$, and for all $x \in X$

$$\lim_{N\to\infty} T_N(t)P_N x = T(t)x$$

uniformly on bounded t-intervals.

The proof of this theorem proceeds in similar steps as that for Theorem 3 and can be found in [14]. If $z = (D(\phi),\phi)$ with $\phi \in C(-r,0;\mathbb{R}^n)$ then $T(t)z = (D(x_t),x_t), t \geq 0$, where $x(t)$ is the solution of the neutral equation

$$\frac{d}{dt} D(x_t) = L(x_t), \qquad t \geq 0,$$

with $x(s) = \phi(s)$, $s \in [-r,0]$. Theorem 4 provides an approximation of $t \to (D(x_t),x_t)$, $t \geq 0$, in X. If we could prove that $T_N(t)P_N z \to T(t)z$ also with respect to the sup-norm, then we would get an approximation of $x(t)$ itself (see [14], Section 6 for details and preliminary results).

c) <u>Equations of Neutral Type in $W^{1,2}(-r,0;\mathbb{R}^n)$</u>

Let the operators D and L be as under b) and put $\tilde{D}(\phi) = \phi(0) - D(\phi)$. The Cauchy problem

$$\dot{x}(t) = L(x_t) + \sum_{j=1}^{m} B_j\dot{x}(t-r_j) + \int_{-r}^{0} B(s)\dot{x}(t+s)ds = L(x_t) + \tilde{D}(\dot{x}_t), \quad t \geq 0,$$

$$x(s) = \phi(s), \ s \in [-r,0],$$

has for any $\phi \in W^{1,2}(-r,0;\mathbb{R}^n)$ a unique solution $x(t;\phi)$ on $[-r,\infty)$. We take $X = W^{1,2}(-r,0;\mathbb{R}^n)$ and define the family $T(\cdot)$ of operators by

$$T(t)\phi = x_t(\phi), \quad t \geq 0, \quad \phi \in W^{1,2}(-r,0;\mathbb{R}^n).$$

$T(\cdot)$ is a C_o-semigroup of bounded linear operators on X with infinitesimal generator A given by

$$\text{dom } A = \{\phi \mid \phi \in W^{2,2}(-r,0;\mathbb{R}^n) \text{ and } \dot{\phi}(0) = \tilde{D}(\dot{\phi}) + L(\phi)\},$$
$$A\phi = \dot{\phi}, \quad \phi \in \text{dom } A.$$

Note, that dom A depends on D and L. We supply X with the norm $|\phi|_g = (|\phi(0)|^2 + \int_{-r}^{0} |\dot{\phi}(s)|^2 g(s)ds)^{1/2}$, where g is as under b) with

$$\alpha_j \leq - \frac{2m^2}{r_j - r_{j-1}} |B_j|^2, \qquad j = 1,\ldots,m.$$

Then there exists an $\omega \in \mathbb{R}$ such that $A - \omega I$ is dissipative in X (cf. [16]).

Theorem 5. Let $\{X_N, P_N, A_N\}$ be an approximation scheme related to A such that:

(i) $\lim_{N\to\infty} P_N\phi = \phi$ for all $\phi \in X$.

(ii) There exists an integer $k \geq 2$ such that for $\phi \in C^k(-r,0;\mathbb{R}^n)$ $\dot{\phi}_N(0) \to \dot{\phi}(0)$ and $\ddot{\phi}_N \to \ddot{\phi}$ in $L^2(-r,0;\mathbb{R}^n)$ as $N \to \infty$, where $\phi_N \in W^{2,2}(-r,0;\mathbb{R}^n)$ is defined by $P_N\phi = \phi_N$.

Then each A_N is a bounded linear operator which generates the C_o-semigroup $T_N(t) = \exp A_N t$ on X and for all $\phi \in X$

$$\lim_{N\to\infty} T_N(t)P_N\phi = T(t)\phi$$

uniformly on bounded t-intervals.

For a proof of this theorem see [16]. Theorem 5 provides, for initial

data in $W^{1,2}(-r,0;\mathbb{R}^n)$, an approximation of x_t with respect to the sup-norm and of $\dot{x}_t$ with respect to the L^2-norm. On the other hand, the subspaces X_N and projection P_N depend on the right-hand side of the equation even if $D(\phi) = \phi(0)$.

4. Spline Approximation

In this section we indicate that subspaces of spline functions can be used in order to get concrete algorithms using the schemes presented in Section 3.

In the situation of Section 3, a) and b) we can take any sequence of subdivisions of $[-r,0]$ with mesh points t_j^N, $j = 1,\ldots,k_N$, such that $\max_j |t_j^N - t_{j-1}^N| / \min_j |t_j^N - t_{j-1}^N| \le \beta < \infty$. In case of Section 3, a) we define X_N to be the subspace of all elements $(\phi(0),\phi)$ such that ϕ is a first order, cubic or cubic Hermite spline with knots at the t_j^N, respectively. We have $\dim X_N = (k_N+1)n$, $(k_N+3)n$ or $2(k_N+1)n$, respectively. Similarly in the case of Section 3, b) X_N is the space of all elements $(D(\phi),\phi)$ with ϕ as above.

In the situation of Section 3, c) we have to satisfy the boundary condition $\dot{\phi}(0) = \tilde{D}(\dot{\phi}) + L(\phi)$ and to guarantee more smoothness. One has to take subdivisions of $[-r,0]$ such that the points $-r_j$, $j = 0,\ldots,m$, are meshpoints and restrict D and L. In the case of cubic splines we have to assume $B_1 = \ldots = B_{m-1} = 0$, $B(s) \equiv A(s) \equiv 0$. Then X_N is the subspace of all cubic splines with knots at the meshpoints t_j^N satisfying $\dot{\phi}(0) = \sum_{j=0}^{m} A_j \phi(-r_j) + B_m \dot{\phi}(-r)$. In the case of cubic Hermite splines we assume

$B(s) \equiv A(s) \equiv 0$ and take X_N to be the subspace of all cubic Hermite splines with knots at the t_j^N such that $\dot{\phi}(0) = \sum_{j=0}^{m} A_j \phi(-r_j) + \sum_{j=1}^{m} B_j \dot{\phi}(-r_j)$.

For details, explicit calculation of bases for X_N, matrix representations of A_N and numerical examples, see [7], [14] and [16].

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